HOMEWORK 2: PROBLEM #5

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**Problem:**

**Consider a slightly damped vibrating string that satisfies:**

**PDE:**

**Boundary Conditions:**

**Initial Conditions:**

Use separation of variables to determine the solutions. You may also assume that .

The first step in separation of variables is to let , so we have separated our solution function into a product of two functions: the spatial function and the temporal function . Now let us find the first and second order derivatives with respect to *x* and *t* for our solution function:

We will substitute (1) - (4) into the PDE to get,

Rearrange the terms and factor we get,

By division of the above line we can separate our temporal and spatial quantities,

From the above extended equality statement we get two ODE’s:

Let us begin with the spatial differential equation. If we let and substitute into the spatial equation we get the following characteristic equation for *r*:

There are three possible cases we need to consider: λ < 0 , λ = 0, λ > 0.

CASE 1) λ < 0 implies so our possible solution will take the form,

We will now consider our boundary conditions to help find the values of A and B. Since and . Therefore for case 1:

Then our spatial function becomes,

Now let us consider,

If we analyze the right half of the equation, , there are two possibilities either

B = 0 or , the latter is impossible unless L = 0 which contradicts the whole premise of having a 1-dimensional string; therefore, the formed B = 0 is true and for the case λ < 0 yields the trivial spatial solution .

CASE 2) λ < 0 implies ; therefore, we have repeated roots and our spatial function may take the form: . We will follow the same steps as in case 1 and use the boundary conditions to determine A and B. For , Now if we consider the other boundary condition: ; therefore, we arrive again at the trivial solution for .

CASE 3) λ > 0 implies ; therefore, we have imaginary roots and our spatial function will take the form: . For the first boundary condition we get: ; therefore, . If we consider the second boundary condition we arrive at a different conclusion: . Here we can conclude that the sine term is only 0 when ; now we have arrived at a relationship for our λ ‘s:

This leads to our eigen-functions:

We will now examine the temporal differential equation:

The characteristic equation for the above differential equation is,

If we utilize the quadratic formula we will obtain the following roots of the equation above,

But remember we may assume that , but notice that our roots, with the eigenvalues incorporated is,

So are these roots real, imaginary? Well let us examine the following,

If then if we square both sides of this inequality we get,

Therefore,

Hence the discriminant of our roots will be negative, we will denote . We will have the following roots of our characteristic equation to the temporal equation:

So we obtain the following solutions to our temporal equation:

So the homogeneous solution to our temporal equation will be a linear combination of our two separate solutions above:

Where,

If we take our spatial solution and temporal we can form our solution to the wave equation,

Then by superposition we have the general series form,

Now we have to find the coefficients utilizing our initial conditions. We will begin with the first initial condition:

If we make the above substitution we get,

We can obtain the coefficients by utilizing the definition of Fourier Series,

Now we will use the second initial condition to find the coefficient. First take the derivative and not we are going to use the product rule with respect to *t*. As you can see the differentiation yields a lengthy equation.

Let t → 0 and we get,

This is again the Fourier Sine series for g(x); therefore,

If we isolate the we get,

But we already know what is, so let us substitute,

Or simply,

Therefore, our final solution to the lightly damped vibrating string is the following,

Where,

It is interesting to note that the depend on f(x) and g(x) and